Direct numerical solution of Burgers equation

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Abstract

In the present study, the Burgers turbulence problem is solved numerically. For this aim, the convective and diffusive terms in the Burgers equation are discretized by applying the weighted essential non-oscillatory (WENO) and the sixth-order central compact difference methods, respectively. The performance of the WENO and sixth-order compact methods are examined by solving the wave and heat equations, respectively. Comparison of the numerical results obtained by these two methods with the analytical ones shows that they can effectively be used for discretizing the first and second-order derivative terms. Indications are that the WENO-compact method can accurately be used for solving the non-linear partial differential equations such as Burgers equation.

I. Introduction

There is a wide range of spatial and temporal scales in the turbulent flows. In order to resolve the full spectra of turbulence, from the large scales to the Kolmogorov scale, a fine mesh should be used in a direct numerical simulation (DNS). In the present study, the Burgers turbulence problem has been simulated by solving the burgers equation using the weighted essential non-oscillatory (WENO) -compact method.

The present study is organized as follows: the governing equation is given in Section II. The spatial discretization including the WENO and the sixth-order compact methods are presented in Section

III, and the temporal discretization is given in Section IV. The boundary conditions are explained in Section V and the numerical results based on the present numerical methodology are presented in Section VI.

II. Governing equation

The burgers equation can be written as follows:

$$\frac{\partial Q}{\partial t}+\frac{\partial f}{\partial x}=\nu\frac{\partial^2 Q}{\partial x^2} \tag{1}$$

where $Q=u$, $f=0.5u^2$, the kinematic viscosity $\nu=5\times10^{-4}$, and u denotes the velocity.

III. Spatial discretization

1. WENO method

The first-order spatial derivative in the Burgers equation (1) can be discretized as follows:

$$\frac{df}{dx}=\frac{1}{\Delta x}\left[\left(f_{j+1/2}^{+}-f_{j-\frac{1}{2}}^{+}\right)+\left(f_{j+1/2}^{-}-f_{j-\frac{1}{2}}^{-}\right)\right] \tag{2}$$

$$\begin{aligned} f_{j+1/2}^{+} &= w_0^{+}\left(\frac{2}{6}f_{j-2}^{+}-\frac{7}{6}f_{j-1}^{+}+\frac{11}{6}f_j^{+}\right)+w_1^{+}\left(-\frac{1}{6}f_{j-1}^{+}+\frac{5}{6}f_j^{+}+\frac{2}{6}f_{j+1}^{+}\right) \\ &\quad + w_2^{+}\left(\frac{2}{6}f_j^{+}+\frac{5}{6}f_{j+1}^{+}-\frac{1}{6}f_{j+2}^{+}\right) \end{aligned} \tag{3}$$

$$\begin{aligned} f_{j-1/2}^{+} &= w_0^{+}\left(\frac{2}{6}f_{j-3}^{+}-\frac{7}{6}f_{j-2}^{+}+\frac{11}{6}f_{j-1}^{+}\right)+w_1^{+}\left(-\frac{1}{6}f_{j-2}^{+}+\frac{5}{6}f_{j-1}^{+}+\frac{2}{6}f_j^{+}\right) \\ &\quad + w_2^{+}\left(\frac{2}{6}f_{j-1}^{+}+\frac{5}{6}f_j^{+}-\frac{1}{6}f_{j+1}^{+}\right) \end{aligned} \tag{4}$$

$$f_{j+1/2}^{-} = w_0^{-}\left(\frac{11}{6}f_{j+1}^{-} - \frac{7}{6}f_{j+2}^{-} + \frac{2}{6}f_{j+3}^{-}\right) + w_1^{-}\left(\frac{2}{6}f_j^{-} + \frac{5}{6}f_{j+1}^{-} - \frac{1}{6}f_{j+2}^{-}\right)$$
$$+ w_2^{-}\left(-\frac{1}{6}f_{j-1}^{-} + \frac{5}{6}f_j^{-} + \frac{2}{6}f_{j+1}^{-}\right) \quad (5)$$

$$f_{j-1/2}^{-} = w_0^{-}\left(\frac{11}{6}f_j^{-} - \frac{7}{6}f_{j+1}^{-} + \frac{2}{6}f_{j+2}^{-}\right) + w_1^{-}\left(\frac{2}{6}f_{j-1}^{-} + \frac{5}{6}f_j^{-} - \frac{1}{6}f_{j+1}^{-}\right)$$
$$+ w_2^{-}\left(-\frac{1}{6}f_{j-2}^{-} + \frac{5}{6}f_{j-1}^{-} + \frac{2}{6}f_j^{-}\right) \quad (6)$$

$$w_j^{\pm} = \frac{\alpha_j^{\pm}}{\sum_{m=0}^{r-1}\alpha_m}, \quad r = 3 \quad (7)$$

$$\alpha_j^{\pm} = \frac{d_j}{\left(\varepsilon + IS_j^{\pm}\right)^2}, \quad \varepsilon = 1 \times 10^{-5}, d_0 = \frac{1}{10}, d_1 = \frac{6}{10}, d_2 = \frac{3}{10} \quad (8)$$

$$IS_0^{+} = \frac{13}{12}\left(f_{j-2}^{+} - 2f_{j-1}^{+} + f_j^{+}\right)^2 + \frac{1}{4}\left(f_{j-2}^{+} - 4f_{j-1}^{+} + 3f_j^{+}\right)^2 \quad (9)$$

$$IS_1^{+} = \frac{13}{12}\left(f_{j-1}^{+} - 2f_j^{+} + f_{j+1}^{+}\right)^2 + \frac{1}{4}\left(f_{j-1}^{+} - f_{j+1}^{+}\right)^2 \quad (10)$$

$$IS_2^{+} = \frac{13}{12}\left(f_j^{+} - 2f_{j+1}^{+} + f_{j+2}^{+}\right)^2 + \frac{1}{4}\left(3f_j^{+} - 4f_{j+1}^{+} + f_{j+2}^{+}\right)^2 \quad (11)$$

$$IS_0^{-} = \frac{13}{12}\left(f_{j+1}^{-} - 2f_{j+2}^{-} + f_{j+3}^{-}\right)^2 + \frac{1}{4}\left(3f_{j+1}^{-} - 4f_{j+2}^{-} + f_{j+3}^{-}\right)^2 \quad (12)$$

$$IS_1^{-} = \frac{13}{12}\left(f_j^{-} - 2f_{j+1}^{-} + f_{j+2}^{-}\right)^2 + \frac{1}{4}\left(f_j^{-} - f_{j+2}^{-}\right)^2 \quad (13)$$

$$IS_2^{-} = \frac{13}{12}\left(f_{j-1}^{-} - 2f_j^{-} + f_{j+1}^{-}\right)^2 + \frac{1}{4}\left(f_{j-1}^{-} - 4f_j^{-} + 3f_{j+1}^{-}\right)^2 \quad (14)$$

$$f^{+} = \frac{1}{2}(f + mQ) \quad (15)$$

$$f^{-} = \frac{1}{2}(f - mQ) \quad (16)$$

where m is the maximum of the absolute eigenvalue entire domain, i.e., $m = \left|\frac{\partial F}{\partial Q}\right|_{max}$. The eigenvalue of the wave and burgers equations are 1 and u, respectively.

2. Sixth-order compact method

The second-order derivatives can be calculated by the sixth-order compact method as

$$\frac{2}{11}Q''_{j-1} + Q''_j + \frac{2}{11}Q''_{j+1} = \frac{12}{11}\frac{Q_{j+1} - 2Q_j + Q_{j-1}}{h^2} + \frac{3}{11}\frac{Q_{j+1} - 2Q_j + Q_{j-1}}{h^2} \quad (17)$$

where h denotes the grid size.

IV. Temporal discretization

The governing equation (1) can be rewritten as:

$$\frac{\partial Q}{\partial t} = R(Q) \quad (18)$$

$$R(Q) = -\frac{\partial f}{\partial x} + \nu\frac{\partial^2 Q}{\partial x^2} \quad (19)$$

where $R(Q)$ denotes the right-hand side vector. There are several methods in order to discretize Eq. (18). Two methods are formulated in this study, namely, the Euler and Fourth-order Runge-Kutta methods.

1. Euler method

The Euler method can be written as follows:

$$Q^{n+1} = Q^n + \Delta t R(Q^n) \quad (20)$$

where Δt is the time step size and the superscript $n+1$ denote the solution at the time $t + \Delta t$.

2. Fourth-order Runge-Kutta method

The Fourth-order Runge-Kutta method can be written as follows:

$$Q^1 = Q^n + \frac{1}{4}\Delta t R(Q^n) \tag{21}$$

$$Q^2 = Q^n + \frac{1}{3}\Delta t R(Q^1) \tag{22}$$

$$Q^3 = Q^n + \frac{1}{2}\Delta t R(Q^2) \tag{23}$$

$$Q^{n+1} = Q^n + \frac{1}{1}\Delta t R(Q^3) \tag{24}$$

V. Boundary equation

For periodic boundary condition, we have

$$JM = 1 \tag{25}$$

$$JM - 1 = 2 \tag{26}$$

$$JM - 2 = 3 \tag{27}$$

where JM denotes the degree of freedom.

VI. Numerical Results

1. The solution of the wave equation using the WENO method

The first-order wave equation can be written as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \tag{28}$$

The initial and boundary conditions considered are

$$u(x,0) = \sin(x)\,, x = [0,2\pi] \tag{29}$$

$$u(0,t) = u(2\pi,t) \tag{30}$$

The analytical solution can be found for this problem as

$$u(x,t) = \sin(x+t) \tag{31}$$

The rate of convergence of the WENO method can be seen in Table 1. As indicated in this table, the rate of convergence is close to 5 which is the optimal rate of convergence. The comparison of the obtained x-velocity profile by applying the WENO method with the analytical one is shown in Fig. 1 and as it can be seen the agreement between results are satisfactory.

Table 1 The rate of convergence for the WENO method

Grid	Error	Rate
20	1.67E-06	4.77572
40	6.10E-08	4.64653
80	2.44E-09	

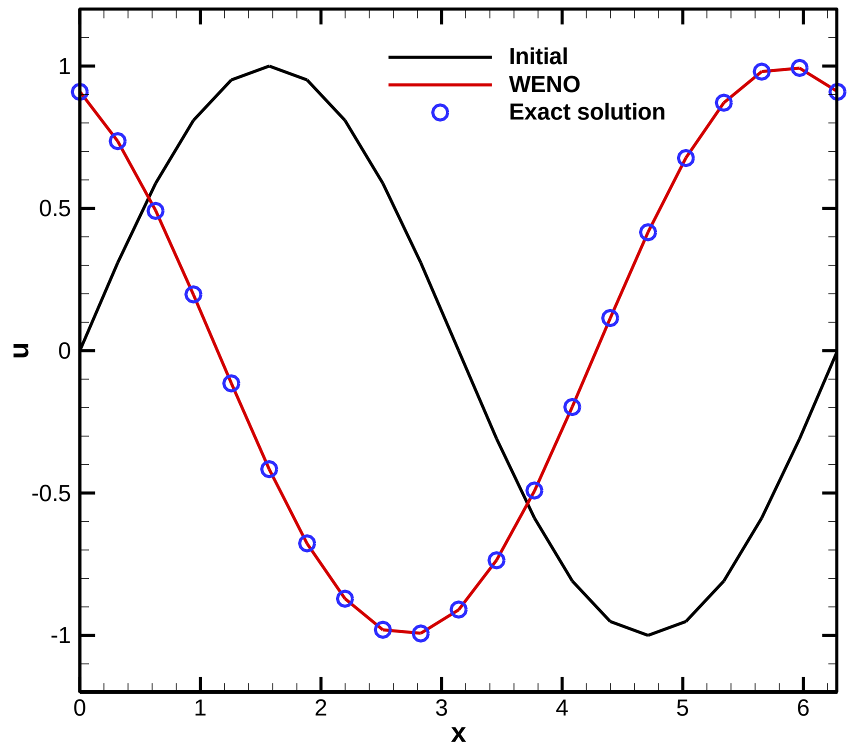

Fig. 1 the x-velocity profile obtained by the WENO method for the wave problem

2. The solution of the heat equation using the compact method

The second-order heat equation can be written as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{32}$$

The initial and boundary conditions considered here are

$$u(x,0) = \sin(\pi x)\,, x = [-1,1] \tag{33}$$

$$u(0,t) = u(2\pi,t) \tag{34}$$

The analytical solution is

$$u(x,t) = \sin(\pi x)\exp(-\pi^2 t) \tag{35}$$

The numerical result and its comparison with the analytical solution is illustrated in Fig. 2 which shows that the compact method approximates accurately the second-order derivatives.

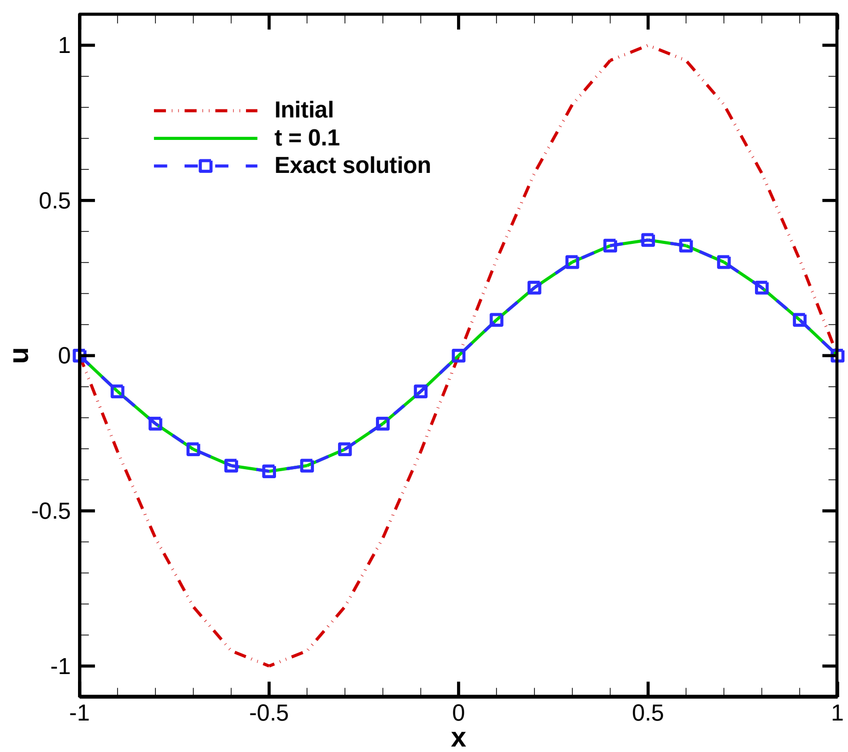

Fig. 2 the x-velocity profile obtained by the compact method for the heat problem

3. Burgers turbulence problem

The initial condition [1] of this problem is given by an initial energy spectrum as follows:

$$E(k) = Ak^4\exp(-(k/k_0)^2) \tag{36}$$

$$A = \frac{2k_0^{-5}}{3\sqrt{\pi}},\quad k_0 = 10 \tag{37}$$

then, the initial velocity in the Fourier space can be obtained as

$$\hat{u}(k) = \sqrt{2E(k)}\exp\big(i2\pi\psi(k)\big) \tag{38}$$

where $\psi(k)$ is a uniform random number distribution between 0 and 1 at each wavenumber. As mentioned in Ref. [1]. This distribution also has to satisfy the $\psi(k) = -\psi(-k)$ conjugate relationship in order to obtain a real velocity field in physical space. Inversions from Fourier space are computed using a Fast Fourier transform algorithm (FFT) given by Press et al. [2].

The numerical results for the energy spectrum and velocity at different times are illustrated in Figures 3 and 4. As obvious in Fig. 3, the energy decays in time. The effects of the viscosity on the velocity profile is shown in Fig. 4. As expected, the amplitude of the velocity speed is decreasing in time. A grid with 512 grid points is used in these two figures.

In Fig. 5, 64 randomly sample fields are constructed with different phases and simulated until $t = 5$ sec and they are indicated by the black line. Ensemble-averaged results for the sample simulations are computed and presented by the red line in this figure. A grid with 512 grid points is used for the results presented in Fig. 5.

A grid study is done to investigate the effect of the grid used on the obtained energy spectrum. As indicated in this figure, the fine grids result in more accurate results.

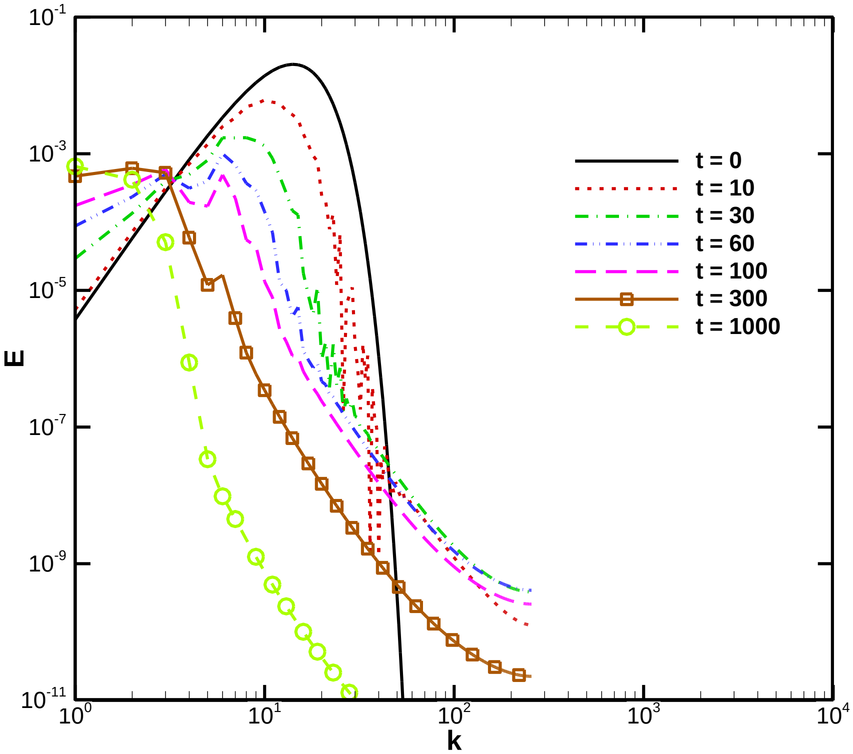

Fig. 3 the energy spectrum profile obtained by the WENO-compact method for the Burgers turbulence problem

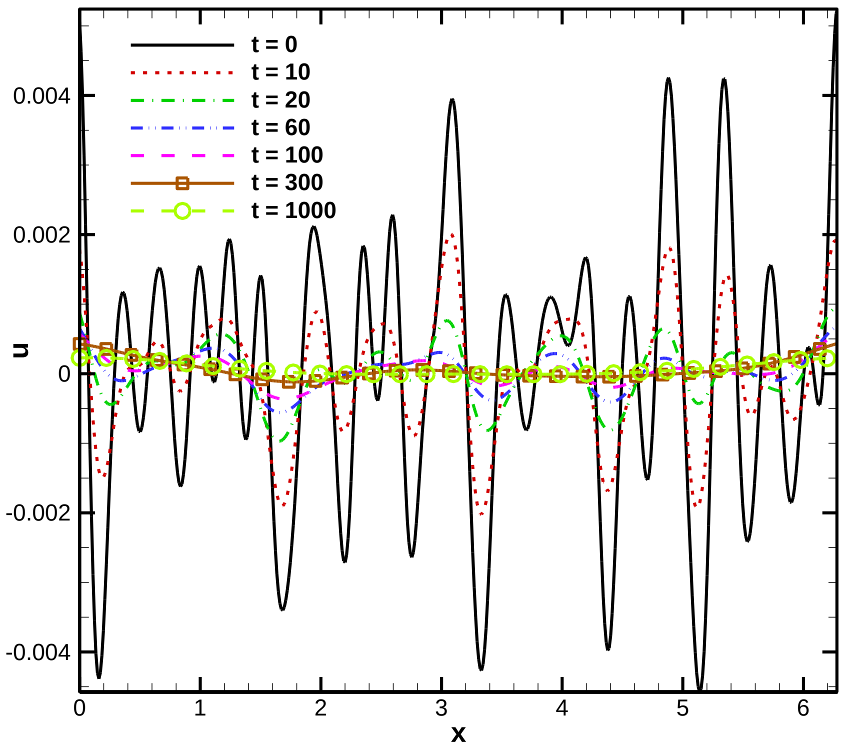

Fig. 4 the velocity profile obtained by the WENO-compact method for the Burgers turbulence problem

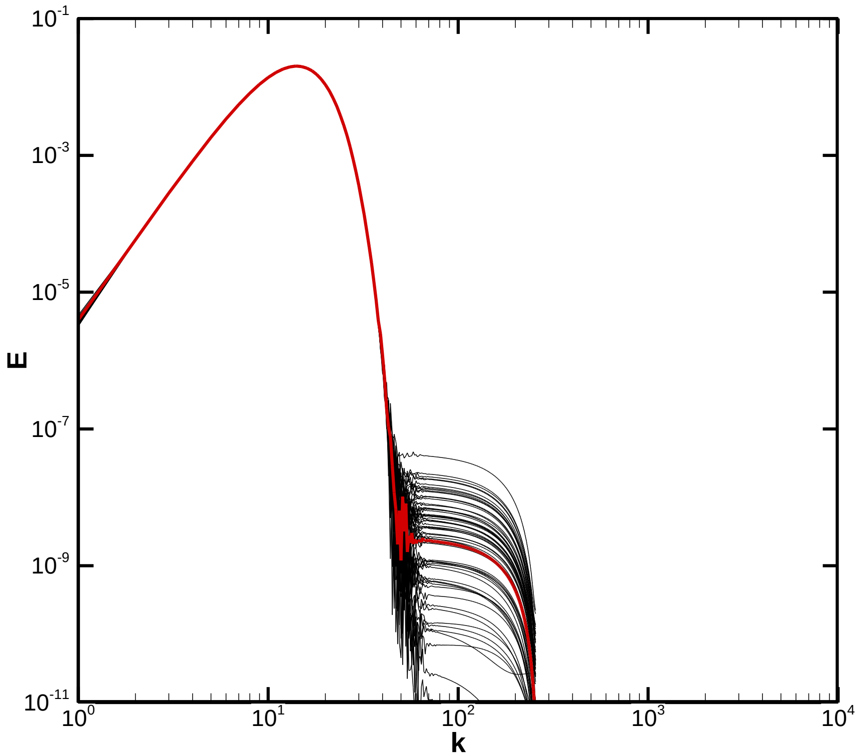

Fig. 5 the Energy spectrum obtained by applying the WENO-compact method to the Burgers equation with 512 grid points. The red line indicates the average of the 64 simulations with different initial phases (the black lines).

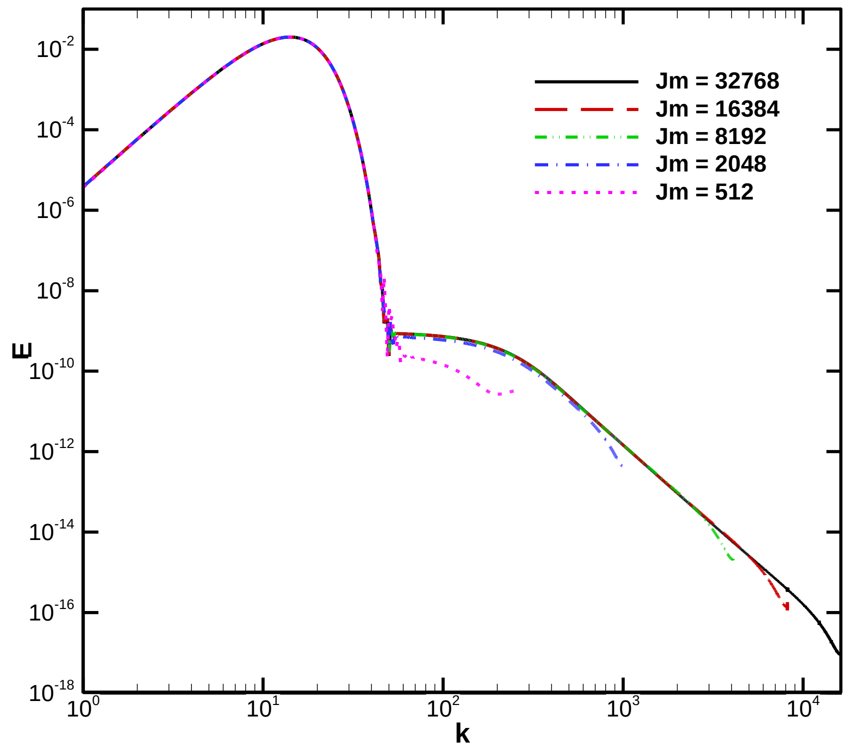

Fig. 6 the grid study for the burgers turbulence problem.

Conclusion

In the present study, the Burgers turbulence problem is solved numerically by utilizing the fifth-order WENO and the sixth-order compact methods for discretizing the first and second-order derivatives, respectively. The wave and heat equations are solved by the WENO and compact methods, respectively, and their accuracy is verified by comparison with the analytical results. Then, the WENO compact method is used for solving the Burgers equation and the energy spectrum and velocity fields are computed. The numerical results show that the present methodology can effectively be used in the direct numerical simulation (DNS) of the burgers turbulence problem. The decay of turbulence is shown in the numerical results and solutions at different times are presented. Indications are that the compact and WENO method can accurately be used for simulating the burgers turbulence problem.

References

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